Transfer Functions, Poles and Zeros

For the design of a control system, it is important to understand how the system of interest behaves and how it responds to different controller designs. The Laplace transform, as discussed in the *Laplace Transforms* module, is a valuable tool that can be used to solve differential equations and obtain the dynamic response of a system. Additionally, the Laplace transform makes it possible to obtain information relating to the qualitative behavior of the system response without actually solving for the dynamic response. The poles and zeros of a system, which are the main focus of this module, provide information on the characteristic terms that will compose the response. This is very useful because it allows a control system designer to understand how the design parameters can be manipulated to obtain acceptable response characteristics. Using a graphical trial and error approach called the root-locus design method, the designer can alter the design parameters to values that lead to an acceptable response and then verify the design by solving for the time response of the system. This module is a continuation of the *Laplace Transforms* module and provides an introduction to the concept of Transfer functions and the poles and zeros of a system.

$with(inttrans)$:

(This command loads the functions required for computing Laplace and Inverse Laplace transforms)

▼ Transfer Functions

A transfer function $G(s)$ is defined as the following relation between the output of the system $Y(s)$ and the input to the system $U(s)$.

$$G(s) = \frac{Y(s)}{U(s)}$$

... Eq. (1)

If the transfer function of a system is known then the response of the system $Y(s)$ can be found by taking the inverse Laplace transform of $G(s) \cdot U(s)$. It is also important to note that a transfer function is only defined for linear time invariant systems with all initial conditions set to zero.

If the input to the system is a unit impulse ($U(s) = 1$), then

$$G(s) = Y(s)$$

<div align="right">... Eq. (2)</div>

Therefore, the inverse Laplace transform of the Transfer function of a system is the unit impulse response of the system. This can be thought of as the response to a brief external disturbance.

▼ Example 1: Transfer function of a Spring-mass system with viscous damping

Problem Statement: The following differential equation is the equation of motion for an ideal spring-mass system with damping and an external force $F(t)$.

$$m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = F(t)$$

Find the transfer function.
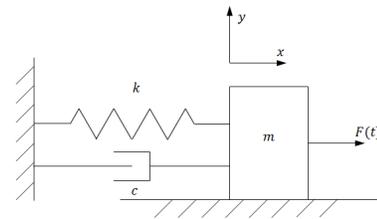
<div align="right">**Fig. 1: Spring-mass system with damping**</div>

▼ Solution

Taking the Laplace transform of both sides of the equation of motion gives

$$m \cdot s^2 \cdot X(s) + c \cdot s \cdot X(s) + k \cdot X(s) = F(s)$$

This equation can be rearranged to get

$$\frac{X(s)}{F(s)} = \frac{1}{m \cdot s^2 + c \cdot s + k}$$

Therefore, the transfer function for this system is

$$G(s) = \frac{1}{m \cdot s^2 + c \cdot s + k}$$

The system response can be found be taking the inverse Laplace transform of $G(s) \cdot F(s)$. If $m = c = k = 1$ and the input is a step function $F(s) = \frac{1}{s}$, then the system response is
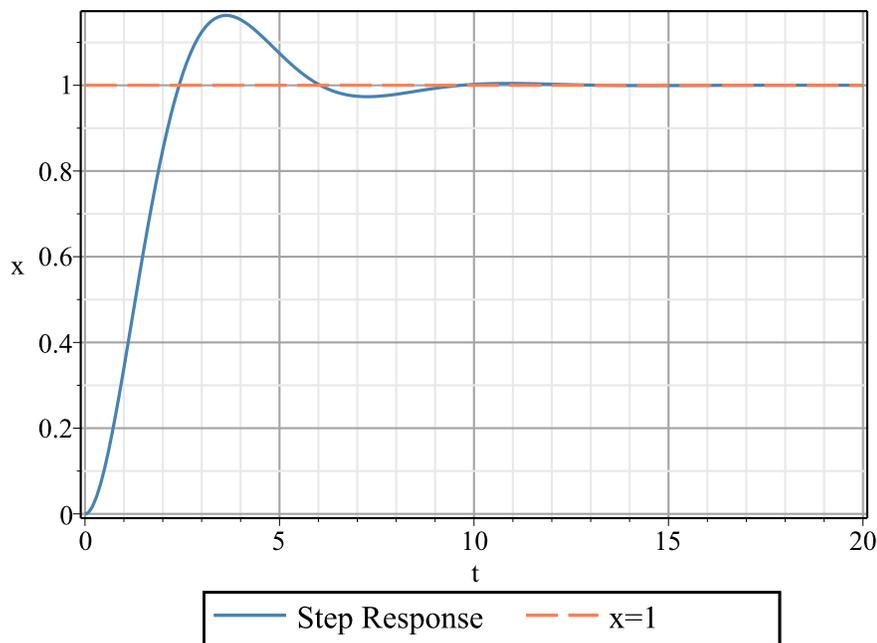
$$x := invlaplace\left(\frac{1}{1 \cdot s^2 + 1 \cdot s + 1} \cdot \frac{1}{s}, s, t \right)$$

$$1 - \frac{1}{3} e^{-\frac{1}{2} t} \left(\sqrt{3} \sin\left(\frac{1}{2} \sqrt{3}\, t \right) + 3 \cos\left(\frac{1}{2} \sqrt{3}\, t \right) \right) \qquad (1.1.1.1)$$

$convert(x, ':-phaseamp', t)$

$$1 - \frac{2}{3} e^{-\frac{1}{2} t} \sqrt{3} \sin\left(\frac{1}{2} \sqrt{3}\, t + \frac{1}{3} \pi \right) \qquad (1.1.1.2)$$

$plot([[(1.1.1.2), 1], t = 0 ..20, color = ["SteelBlue", "Coral"], linestyle = [solid, dash], labels = ["t", "x"], legend = ["Step Response \quad ", "x=1"], axes = boxed, gridlines = true);$

Example 2: Transfer function of a DC Motor (with MapleSim)

Problem Statement: A DC motor is modeled using the equivalent circuit shown in Fig. 2.

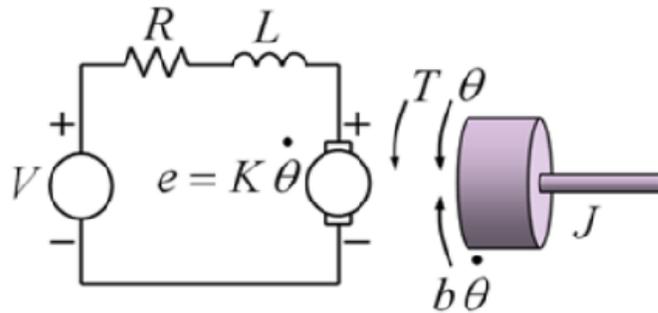Find the transfer function relating the angular velocity of the shaft and the input voltage.

Fig. 2: DC Motor model

This example demonstrates how to obtain the transfer function of a system using MapleSim.

Analytical Solution

The equivalent circuit consists of a voltage source which is the input, a resistor, an inductor and a "back EMF" voltage source. The back EMF depends on the rate of rotation and can be expressed as

$$e = K_e \cdot \frac{\mathrm{d}}{\mathrm{d}\,t} \theta(t)$$

where K_e is a constant of proportionality called the electric constant and $\frac{\mathrm{d}}{\mathrm{d}\,t} \theta(t)$ is the angular speed.

The torque on the rotor is proportional to the armature current $i(t)$ and can be expressed as

$$T = K_t \cdot i(t)$$

where K_t is a constant of proportionality called the torque constant. It should be noted that the electric constant and the motor constant are equal to each other when

expressed in the same units ($K = K_e = K_t$).

The dynamic equation for the circuit is

$$V(t) = i(t) \cdot R + L \cdot \frac{d}{dt} i(t) + K \cdot \frac{d}{dt} \theta(t)$$

where $V(t)$ is the input voltage, R is the resistance of the resistor and L is the inductance of the inductor. The Laplace transform of this equation is

$$V(s) = (R + s \cdot L) \cdot I(s) + K \cdot s \cdot \Theta(s)$$

... Eq. (3)

The dynamic equation for the rotor is

$$K \cdot i(t) = J \cdot \frac{d^2}{dt^2} \theta(t) + b \cdot \frac{d}{dt} \theta(t)$$

where J is the moment of inertia of the rotor and b is the damping constant. The Laplace transform of this equation is

$$K \cdot I(s) = J \cdot s^2 \cdot \Theta(s) + b \cdot s \cdot \Theta(s)$$

... Eq. (4)

Combining Eqs. (3) and (4) and eliminating $I(s)$ yields

$$V(s) = (R + s \cdot L) \cdot \left(\frac{J \cdot s^2 \cdot \Theta(s) + b \cdot s \cdot \Theta(s)}{K} \right) + K \cdot s \cdot \Theta(s)$$

This equation can be rearranged to obtain the required transfer function:

$$\frac{\Theta(s)}{V(s)} = \frac{K}{(J \cdot L) \cdot s^3 + (J \cdot R + L \cdot b) \cdot s^2 + (R \cdot b + K^2) \cdot s}$$
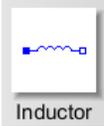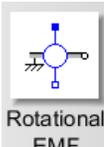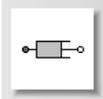
Solution using MapleSim

Constructing the model

Step 1: Insert Component

Drag the following components into the workspace:

Table 1: Components and locations

Component	Location
Step	Signal Blocks > Common
Signal Voltage	Electrical > Analog > Sources > Voltage
Resistor	Electrical > Analog > Common
Inductor	Electrical > Analog > Common
Rotational EMF	Electrical > Analog > Common
	Electrical >
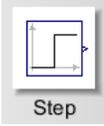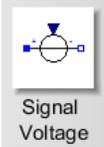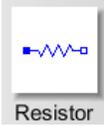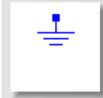
Ground	Analog > Common
Rotational Damper	1-D Mechanical > Rotational > Common
Inertia	1-D Mechanical > Rotational > Common
Rotational Fixed	1-D Mechanical > Rotational > Common
Angle Sensor	1-D Mechanical > Rotational > Sensors

Step 2: Connect the components

Connect the components as shown in the following diagram:
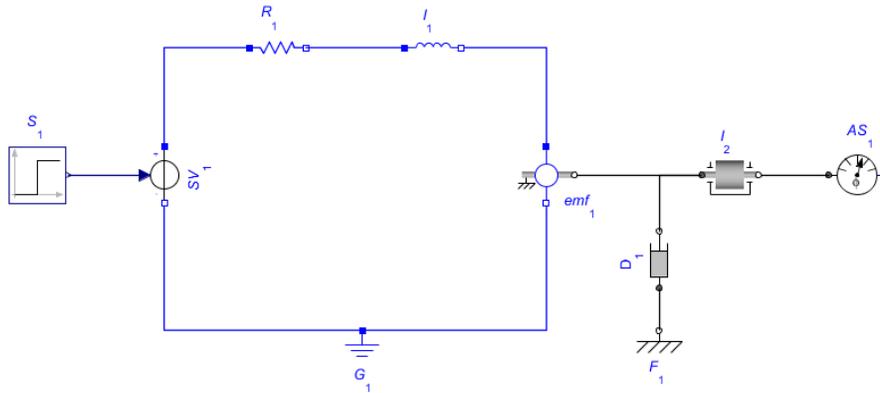
Fig. 3: Component diagram

Step 3: Create a subsystem

1. Highlight all the components, excluding the **Step** component.
2. Press **Ctrl+G** to create the subsystem.
3. Name the subsystem **DCMotor** and click **OK**.
4. Double click the subsystem and click **Add or Change Parameters** in the inspector tab.
5. Create the parameters as shown below.

DCMotor subsystem default settings

Name	Type		Default Value	Default Units	Description
R	Real	▼	1		Resistance
L	Real	▼	1		Inductance
J	Real	▼	1		Moment of Inertia
k	Real	▼	1		Back emf constant
b	Real	▼	1		Damping constant

Fig. 4: Parameters

6. Return to the subsystem component diagram and enter these variables for the corresponding parameters of the components. For example, click the **Resistor** component and enter R for the **Resistance (R)** in the **Inspector** tab.
7. Click the output of the **Angle Sensor** component and connect it to the dashed line that represents the boundary of the subsystem.
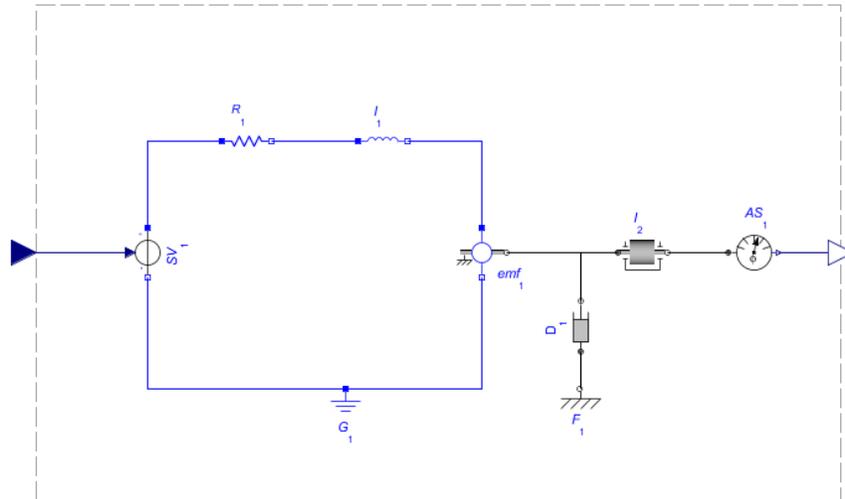
Fig. 5: Subsystem

▼ Obtaining the system equations

1. Click the **Create attachment from template** icon (📎).

2. Select **Equations** from the list and click **Create Attachment**. This will launch a Maple window.

3. In the launched worksheet, select the **DCMotor** subsystem in the drop-down menu for **Step 1: Subsystem Selection** and then click **Load Selected Subsystem**.

4. Under **DAE Variables** rename the variables to simplify the equations. Rename **I2_phi(t), I2_w(t),SV1_n_v(t), emf1_p_i(t)** and **u1(t)** as **phi(t), w(t), v(t), i(t)** and **u(t)** respectively.

5. Click **Reassign Equations**.

6. Scroll down to **Step 2: View Equations**. These are the dynamic equations for the subsystem and are assigned to the variable **DAEs**.

7. Scroll down to the bottom of the work sheet and execute the following commands to obtain the system transfer function.

► Equation Manipulation

$sys := DynamicSystems[TransferFunction](DAEs, [u(t)], [phi(t)]);$

$$\begin{bmatrix} \textbf{Transfer Function} \\ \text{continuous} \\ \text{1 output(s); 1 input(s)} \\ \text{inputvariable} = [u(s)] \\ \text{outputvariable} = [\phi(s)] \end{bmatrix}$$ (1)

$sys{:}\text{-}tf;$

$$\left[\frac{k}{JL\,s^3 + (JR + bL)\,s^2 + (k^2 + Rb)\,s} \right]$$ (2)

Fig. 6: System Transfer Function

This transfer function matches the one obtained analytically.

▼ Poles and Zeros

Zeros are defined as the roots of the polynomial of the numerator of a transfer function and *poles* are defined as the roots of the denominator of a transfer function. For the generalized transfer function

$$G(s) = \frac{(s - z_1) \cdot (s - z_2) \ldots \cdot (s - z_n)}{(s - p_1) \cdot (s - p_2) \ldots \cdot (s - p_n)}$$

... Eq. (5)

The zeros are $z_1, z_2, \ldots z_n$ and the poles are $p_1, p_2, \ldots p_n$.

Identifying the poles and zeros of a transfer function aids in understanding the behavior of the system. For example, consider the transfer function $F_1(s) = \dfrac{32}{s \cdot (s + 1) \cdot (s + 2)}$ .This function has three poles, two of which are negative integers and one of which is zero. Using the method of partial fractions, this transfer function can be written as $F_1(s) = \dfrac{16}{s} - \dfrac{32}{s + 1} + \dfrac{16}{s + 2}$ and its time response (with a unit impulse input) can be found to be $f_1(t) = 16 - 32 \cdot e^{-t} + 16 \cdot e^{-2 \cdot t}$. This shows that the negative poles contribute exponential

terms that decay with time and that the pole at 0 contributes a constant term. If we take another transfer function, for example $H_1(s) = \dfrac{2}{s \cdot (s - 1) \cdot (s + 1)}$, without solving for the solution, we can now conclude that the pole at 0 will contribute a constant term, the negative pole will contribute a term that decays with time and the positive pole will contribute a term that grows with time. This allows us to further conclude that the response will be unstable because it will continuously grow with time due to the positive pole. The following plot shows the time response of $H_1(s)$.

Response Plot
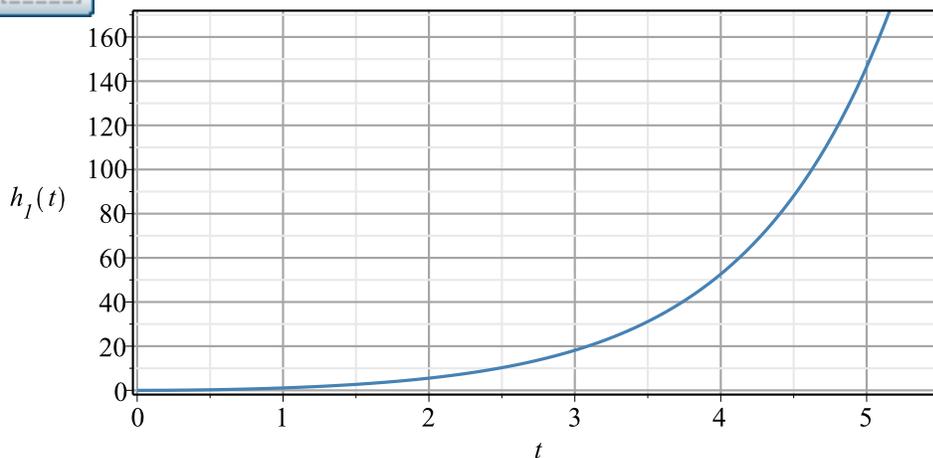
Now consider the transfer function $F_2(s) = \dfrac{16}{s \cdot (s + 1 - j) \cdot (s + 1 + j)}$. This function also has three poles, however, two of these are complex. Using the method of partial fractions, this can be written as $F_2(s) = \dfrac{8}{s} - \dfrac{4 \cdot (1 - j)}{(s + 1 - j)} - \dfrac{4 \cdot (1 + j)}{(s + 1 + j)}$ and the time response (with a unit impulse input) can be found to be $f_2(t) = 8 - 8 \cdot e^{-t} \cdot [\cos(t) + \sin(t)]$. This shows that the complex poles contribute sinusoidal terms and result in oscillations in the system response.

These examples illustrate that the location of the poles on a complex plane can help obtain a qualitative understanding of characteristics of the time response. The following plot shows the poles of the transfer functions of $F_1(s)$ and $F_2(s)$ plotted on the complex plane (or the s-plane).

with(*DynamicSystems*) :

$$ZeroPolePlot\left(TransferFunction\left(\left\langle \frac{32}{s\cdot(s+1)\cdot(s+2)} \middle| \frac{16}{s\cdot(s+1-j)\cdot(s+1+j)} \right\rangle \right), color = [red,$$

$$blue], labels = [\text{"Real axis"}, \text{"Imaginary axis"}], legend = [F_1(s)\ Poles, F_2(s)\ Poles], symbolsize$$

$$= 30 \right)$$
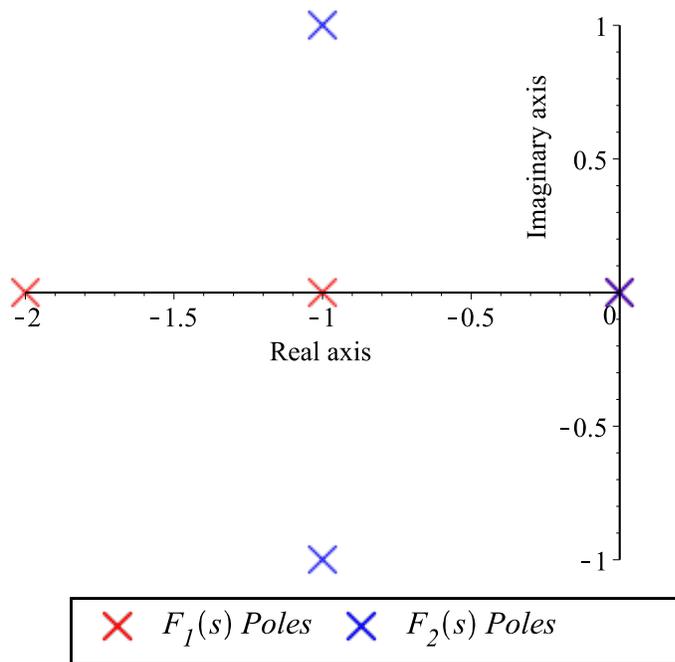
The interactive plots given below can be used to better understand the effect of pole locations on a system's response. The following plot shows the transient response of a system with two real poles for a unit-impulse input and a unit-step input. One of these poles is fixed at -0.5 and the other can be dragged on the real axis to see the effect on the response.

with(*plots*) :
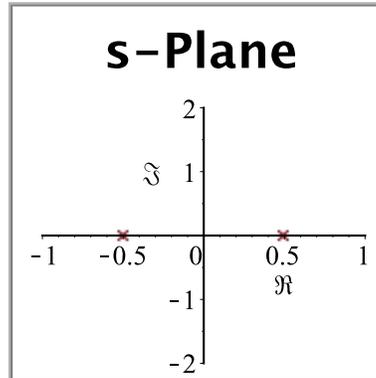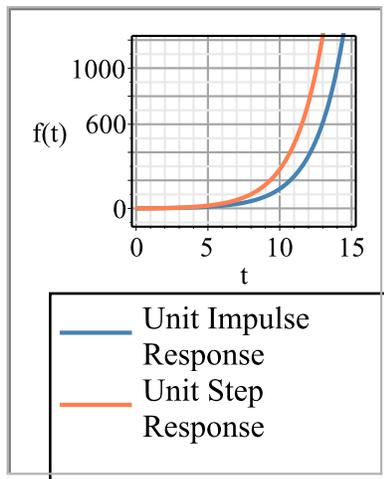
The following plot shows the transient response of a system with a real pole and a pair of complex poles for a unit-impulse input and a unit-step input. These poles can be dragged on the s-plane to see the effect on the response.
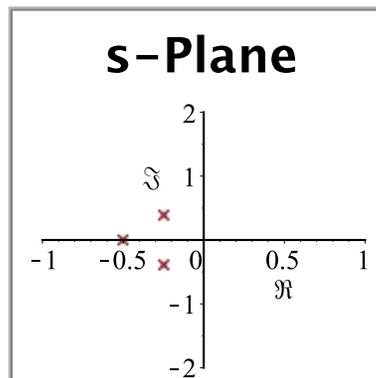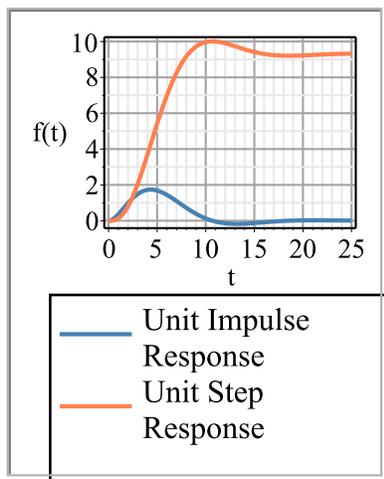
The following plot shows the transient response of a system with a real zero and a pair of complex poles for a unit-impulse input and a unit-step input. The response of the system without the zero is also included for comparison. The poles and zero can be dragged on the s-plane to see the effect on the response.

Unit Impulse Response	
Unit Step Response	
Unit Impulse Response without the zero	
Unit Step Response without the zero	

s-Plane
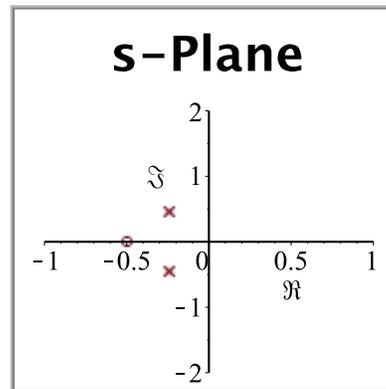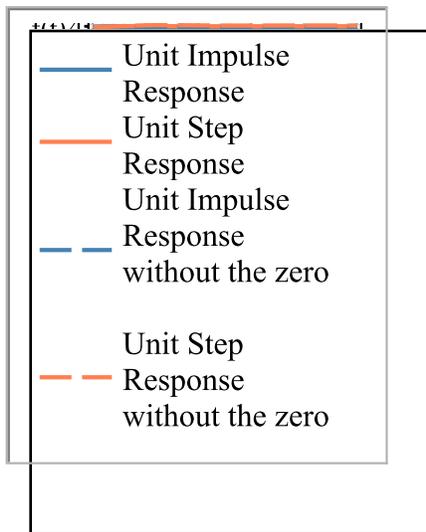
The effect of zeros are not covered in detail in this module; however, it is important to note that the step response of a system with a pole is a combination of a step and an impulse response of the system without the pole:

The step response of the transfer function $G(s) = \dfrac{(s - z_1)}{(s - p_1)\cdot(s - p_2)\,...\,(s - p_n)}$ can be written as

$$G(s) = \frac{1}{s}\cdot\left[\frac{(s - z_1)}{(s - p_1)\cdot(s - p_2)\,...\,(s - p_n)}\right]$$

This can be expanded to get

$$G(s) = \frac{1}{(s - p_1)\cdot(s - p_2)\,...\,(s - p_n)} - \frac{1}{s}\cdot\frac{z_1}{(s - p_1)\cdot(s - p_2)\,...\,(s - p_n)}$$

The first term on the RHS is an impulse response and second term is a step response.

▼ Unit impulse response plots for some different cases

This subsection contains some more plots that show the effect of pole locations and help illustrate the general trends.

Dynamic response plots	Transfer function pole locations

Comparison of:

$$F(s) = \frac{1}{(s+1)}, \; G(s) = \frac{1}{(s+2)} \; \& \, H(s) = \frac{1}{(s+3)}$$

.

Response plot

Poles plot

- ■ F(s) ■ G(s)
- ■ H(s)

Comparison of:

$$F(s) = \frac{1}{(s-1)}, \; G(s) = \frac{1}{(s-2)} \; \& \, H(s) = \frac{1}{(s-3)}$$
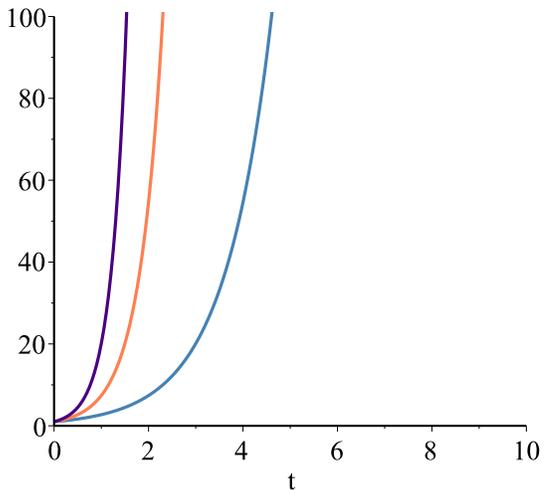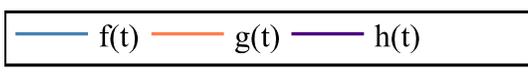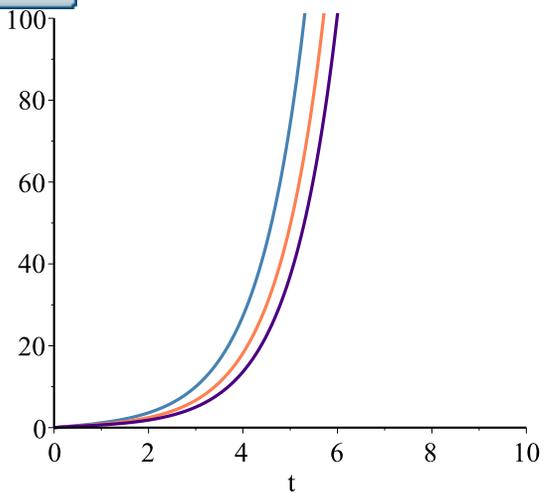
.
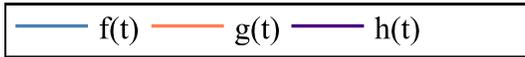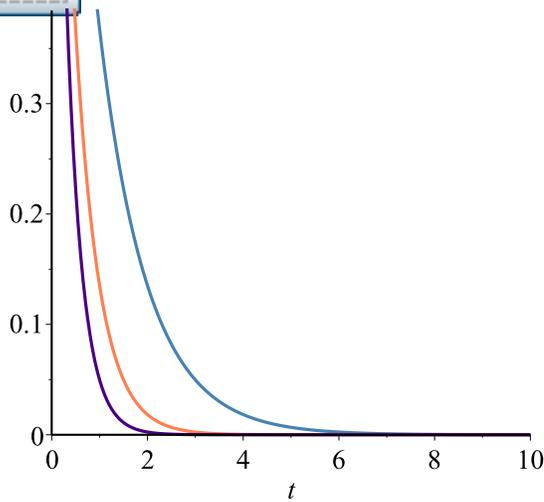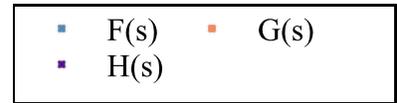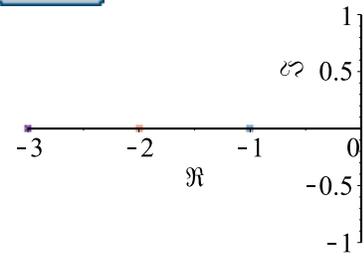
Response Plot
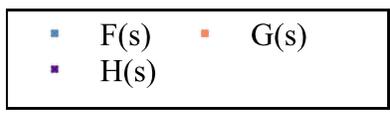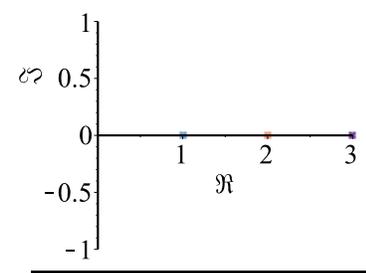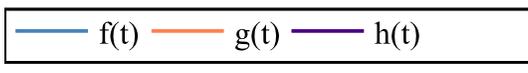
Poles plot

Comparison of:

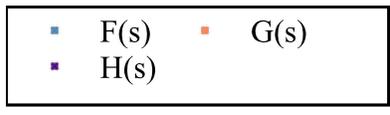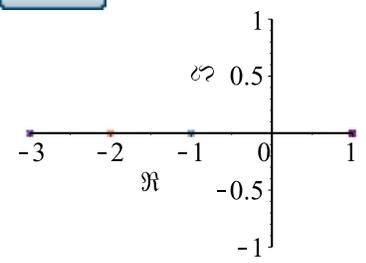$$F(s) = \frac{1}{(s+1)\cdot(s-1)},\, G(s) = \frac{1}{(s+2)\cdot(s-1)}$$

$$\&\, H(s) = \frac{1}{(s+3)\cdot(s-1)}\ .$$

Response Plot

Poles plot

Comparison of:

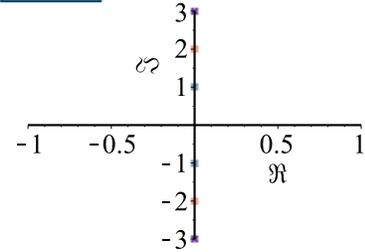$$F(s) = \frac{1}{(s+j)\cdot(s-j)}, \quad G(s) = \frac{1}{(s+2\cdot j)\cdot(s-2\cdot j)}$$

$$\& H(s) = \frac{1}{(s+3\cdot j)\cdot(s-3\cdot j)} \ .$$

Response Plot
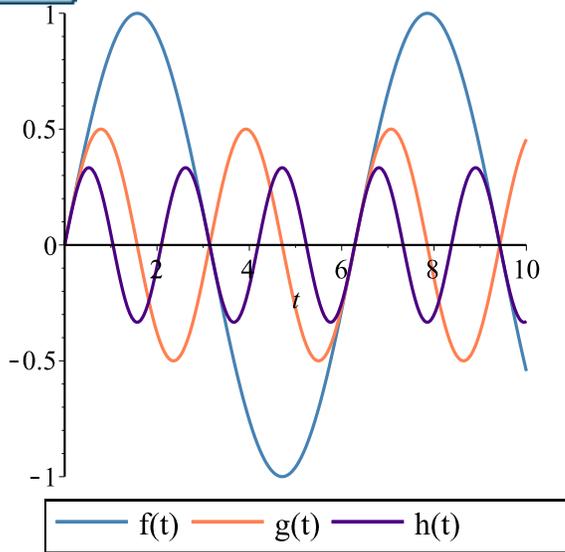
Poles plot

Comparison of:

$$F(s) = \frac{1}{(s+j)\cdot(s-j)}, \quad G(s)$$
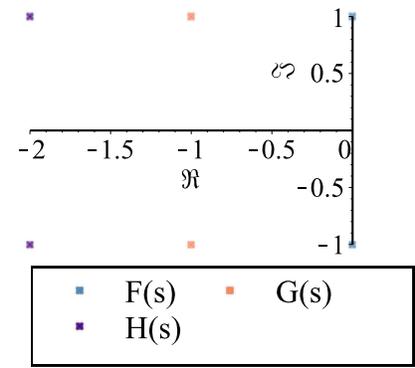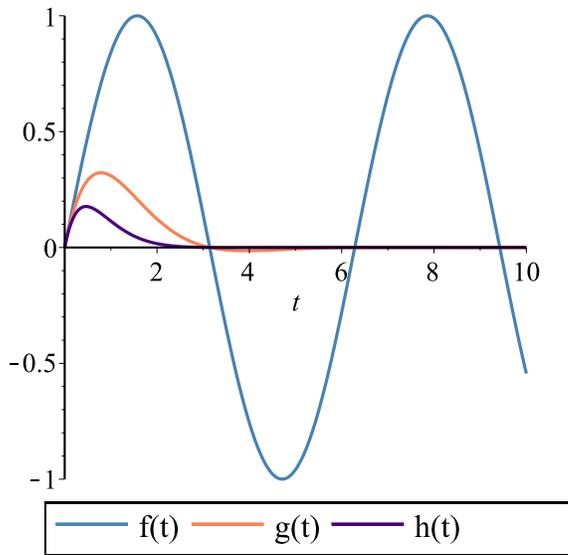
$$= \frac{1}{(s+1+j)\cdot(s+1-j)} \ \& H(s)$$

$$= \frac{1}{(s+2+j)\cdot(s+2-j)} \ .$$
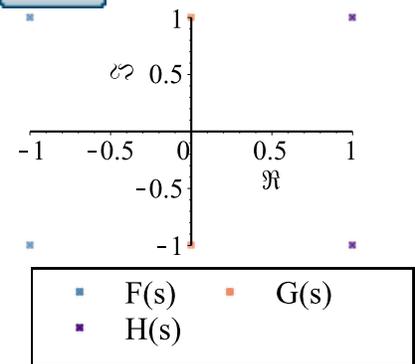
Response Plot

Poles plot

Comparison of:

$$F(s) = \frac{1}{(s+1+j)\cdot(s+1-j)}, G(s)$$

$$= \frac{1}{(s+j)\cdot(s-j)} \& H(s)$$

$$= \frac{1}{(s-1+j)\cdot(s-1-j)}.$$
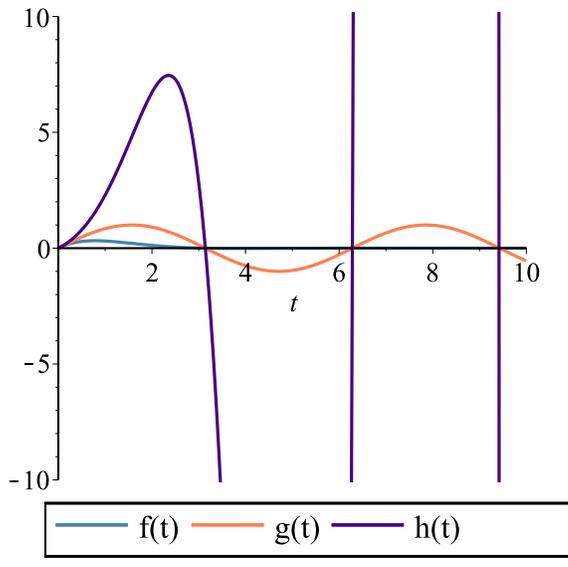
 Response Plot

 Poles plot

Fig. 7 shows the general rule of how the location of the poles on the s-plane effects the time response of a system.
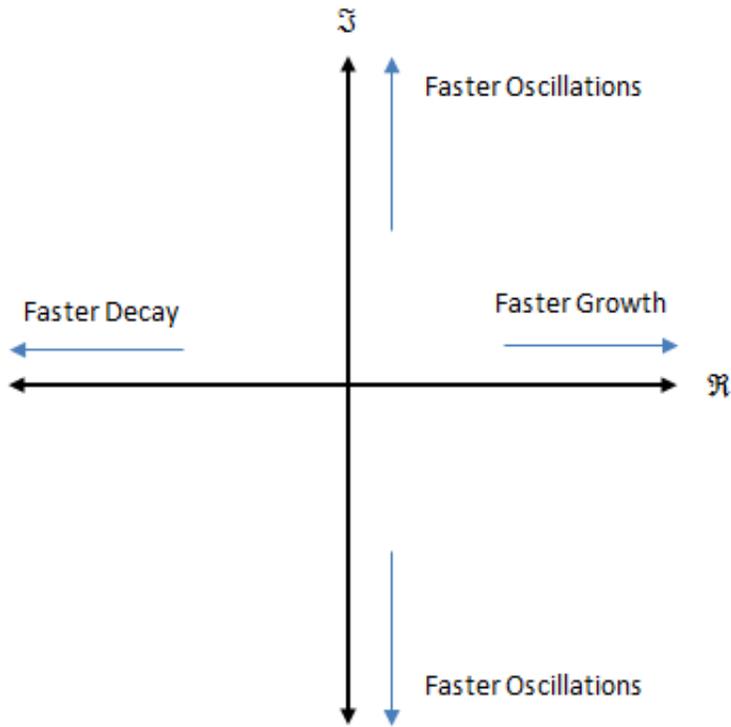
Fig. 7: The s-plane

References:

1. G.F. Franklin et al. "Feedback Control of Dynamic Systems", 5th Edition. Upper Saddle River, NJ, 2006, Pearson Education, Inc.
2. D. J. Inman. "Engineering Vibration", 3rd Edition. Upper Saddle River, NJ, 2008, Pearson Education, Inc.